FINAL EXAM ISI BANGALORE 27TH NOV 2012

INTRODUCTION TO DIFFERENTIAL GEOMETRY 100 POINTS

Notes.
(a) Begin each answer on a separate sheet.
(b) Justify your steps. Unless mentioned otherwise, assume only those theorems that have been proved in class. All other steps should be justified.
(c) You have a total of 125 points to attempt from but you will at the most be awarded 100.

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0. [10 points] These marks are for writing neatly, coherently etc. Be concise but don't skip steps. Finish within 3 hours.

1. [15 points] Let $\vec{\sigma}\colon U \to \mathbb{R}^3$ be a regular surface patch of a surface S sending the origin $(0,0) \in U \subset \mathbb{R}^2$ to the origin $p = (0,0,0) \in S$. For any plane H through p which is distinct from the tangent plane at p, prove that $H \cap S$ is a regular parametrized curve in some neighborhood of p.

2. [15 points] Let $\vec{\gamma}(s)$ denote a unit-speed curve in $\mathbb{R}^3$. Suppose its curvature and torsion are given by $\kappa(s) = s$, $\tau(s) = s^2$ respectively ($s > 0$). If $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ denote the unit tangent, normal and binormal vectors respectively at each point, express $\ddot{\vec{t}}$ as a linear combination of these three vectors at each point.

3. [20 points] Let $f, g \in C^\infty(\mathbb{R}^2)$. Assume that the surface patch $\vec{\sigma}(u,v) = (f(u,v), g(u,v), 0)$ (of the XY-plane in $\mathbb{R}^3$) is regular over some open subset $U \subset \mathbb{R}^2$.

   (i) Prove that $\vec{\sigma}$ induces a conformal equivalence on U (i.e., the first fundamental form via $\vec{\sigma}$ is conformally equivalent to the standard metric on $\mathbb{R}^2$) iff the following two equations are satisfied in U

   $$f_u^2 + g_u^2 = f_v^2 + g_v^2, \qquad f_u f_v + g_u g_v = 0.$$

   (ii) Assume that U is connected. Prove that either $f_u = g_v$ and $f_v = -g_u$ hold everywhere or $f_u = -g_v$ and $f_v = g_u$ hold everywhere. (Aside: These are the Cauchy-Riemann equations and their anti-holomorphic counterpart respectively.)

4. [20 points] Let S be the surface defined by $X^2+2Y+3Z^2=0$. Using the basic definitions only, compute the second fundamental form, the principal curvatures, the principal tangent vectors and the Gaussian curvature of S at the origin.

5. [15 points] Let C be a circle of radius r on the unit sphere. (Here C is the intersection of the unit sphere with some plane.) Compute the absolute values of the normal and geodesic curvature of C at any point in terms of r.

6. [15 points] Recall that the pseudosphere is given by the surface of revolution

$$\vec{\sigma}(u,v) = (e^u \cos v, e^u \sin v, g(u))$$

where g can be determined by the equation $\dot{g}^2 = 1 - e^{2u}$ and $u < 0, -\pi < v < \pi$. Let us reparametrize the pseudosphere as $\vec{\psi}(v,w) = \vec{\sigma} \circ F$ where $F(v,w) = (-\ln w, v)$ is a diffeomorphism defined for $w > 1, -\pi < v < \pi$. (We are merely substituting $w = e^{-u}$ and interchanging the order in which we write the variables). Deduce that the first fundamental form via ψ is

$$\frac{(dv)^2 + (dw)^2}{w^2}.$$

7. [15 points] Recall that Archimedes' theorem says that the cylindrical projection for the unit sphere is equi-areal.

(i) Use Archimedes's theorem to prove that the area of the region on the unit sphere trapped between two parallel planes of distance d apart (and each intersecting the sphere) equals $2\pi d$ and thus depends only on d and not on their positions vis-a-vis the sphere.

(ii) Use this to give a short solution to the following puzzle:
Suppose the unit disc in $\mathbb{R}^2$ is covered by finitely many strips S_i of width d_i. Prove that $\Sigma_i d_i \geq 2$.
(Here a strip of width d means the region trapped between two parallel lines of distance d apart.)

(Take-home exercise: Convince yourself that if equality in (ii) holds then all the S_i's are aligned in the same direction, i.e., are given by lines of same slope, and any two S_i's intersect at most along their boundary line.)